

# Blind Blur Estimation Using Low Rank Approximation of Cepstrum

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**Abstract.** The quality of image restoration from degraded images is highly dependent upon a reliable estimate of blur. This paper proposes a blind blur estimation technique based on the low rank approximation of cepstrum. The key idea that this paper presents is that the blur functions usually have low ranks when compared with ranks of real images and can be estimated from cepstrum of degraded images. We extend this idea and propose a general framework for estimation of any type of blur. We show that the proposed technique can correctly estimate commonly used blur types both in noiseless and noisy cases. Experimental results for a wide variety of conditions i.e., when images have low resolution, large blur support, and low signal-to-noise ratio, have been presented to validate our proposed method.

## 1 Introduction

The first and foremost step in any image restoration technique is blur estimation which is referred to as blind blur estimation when partial or no information about imaging system is known. Numerous techniques [1] have been proposed over the years which try to estimate Point Spread Function (PSF) of blur either separately from the image [2, 3, 4, 5], or simultaneously with the image [6, 7]. The first set of these techniques, also known as Fourier-based techniques, normally use the idea of finding zeros in the frequency domain. These techniques are widely popular because these tend to be computationally simple and require minimal assumptions on images. But these techniques require small blur support or relatively high signal-to-noise ratio. A lot of past research has focused on these requirements and how to relax them. One such technique [2] proposed reducing the effect of noise by using an averaging scheme over smaller portions of the image, thus requiring the original image to be of high resolution. Another technique [8] suggested that the bispectrum of the ‘central slice’ should be used in order to estimate the blur. In that work, only motion blur with small support was estimated at low signal-to-noise ratios. Another related approach was proposed in [9], where image restoration through operations on singular vectors and singular

values of degraded image was proposed. In their work several images were used to remove the effect of noise. Thus far, no single Fourier-based technique has been able to estimate common blur functions correctly from *single* degraded image while ensuring that the technique works not only for various amounts of blur support in low resolution images but also at low signal-to-noise ratios.

In this work, we address the blind blur estimation problem for images that may not be of high resolution and may have noise present during the blurring process. We estimate the blur parameters using low rank constraints on blur functions without performing any operations on singular vectors directly. Our technique assumes that the type of blur added is known but no other assumptions on any prior knowledge about the original image is made. We observe that the blurs are usually low rank functions and thus can be separated from images of high ranks. Based on this observation, we quantify the exact rank approximation for different blur types. These low rank constraints, when applied to log-power spectrum (the power cepstrum) of degraded image, allow us to estimate the blur functions. We also demonstrate that this technique works for a wide variety of blur types (e.g., Gaussian, Motion and Out of focus blurs), when blur support is not small and also when additive noise is present. Experimental results for real images (both noiseless and noisy cases) are presented for various signal-to-noise ratios.

The rest of the paper is organized as follows: Section 2 reviews general concepts of low rank approximation and image degradation and also introduces different blur models. Section 3 presents means to estimate blur parameters using low rank approximation of cepstrum whereas results for different blur types in several scenarios are presented in Section 4. Section 5 presents conclusion of this work.

## 2 Theory

### 2.1 Low Rank Approximation

Low rank approximation is a well known tool for dimension reduction and has been widely used in many areas of research. For a given data  $A$ , its Singular Value Decomposition (SVD) can be written as,

$$A = U \Sigma V^T$$

where  $U$  and  $V$  are orthogonal and  $\Sigma$  is diagonal.  $A$  can be approximated as,

$$A \approx U_k \Sigma_k V_k^T \tag{1}$$

where  $k$  is the rank that best approximates  $A$ ;  $U_k$  and  $V_k$  are the matrices formed by the first  $k$  columns of the matrices  $U$  and  $V$  respectively; and  $\Sigma_k$  is the  $k$ -th principal submatrix of  $\Sigma$ . We use this formulation along with an observation, that blurs are usually low rank functions, to quantify their exact rank approximation in Section 3.

## 2.2 Image Degradation and Blur Models

The goal in a general blind blur estimation problem is, to separate two convolved signals,  $\mathbf{i}$  and  $\mathbf{b}$ , when both are either unknown or partially known. The image degradation process is mathematically represented by convolution of image  $\mathbf{i}$  and blur  $\mathbf{b}$  in time domain i.e.,  $h(x, y) = i(x, y) * b(x, y)$ . The frequency domain representation of this system is given by point-by-point multiplication of image spectrum with blur spectrum.

$$H(m, n) = I(m, n)B(m, n) \quad (2)$$

The system in (2) depicts a noiseless case when image degradation occurred only due to the blur function. The noisy counterpart of such degradation can be modelled as,

$$H(m, n) = I(m, n)B(m, n) + N(m, n) \quad (3)$$

where  $N(m, n)$  denotes the spectrum of an additive noise.

A list of optical transfer functions (OTF) of frequently used blurs has been given in Table 1. It should be noted that OTF is the Fourier transform version of PSF. In the experiments presented in this paper, we have estimated OTFs of blur functions.

**Table 1.** Frequently used blur models

Blur Model	Rank of Blur	Optical Transfer Function (OTF)
Linear Motion Blur	1	$\frac{\sin(\pi a f_x)}{\pi a f_x}$
Gaussian Blur	1	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(f-\mu)^2}{2\sigma^2}}$
Out of focus Blur	10	$\frac{J_1(2\pi r f)}{\pi r f}$

## 3 Blur Estimation Using Low Rank Approximation of Cepstrum

The logarithm of the absolute spectra of degraded image (also known as power cepstrum) can be written as the summation of logarithms of absolute spectra of image and blur as in (4). We propose that these can be separated when low rank constraints are applied. Table 1 summarizes ranks for commonly used blur functions<sup>1</sup>.

Power-Cepstrum of (2), can be written as,

$$H_p = I_p + B_p \quad (4)$$

<sup>1</sup> Exact value of rank for out of focus blur depends on the value of blur parameter.

where  $H_p$  is the log-power spectrum or power-cepstrum of  $h(x, y)$ . It should be noted that  $H_p$  is the summation of two matrices  $I_p$  and  $B_p$  where  $B_p$  has low rank and  $I_p$  has high rank. Therefore, by performing low rank approximation using SVD of (4), we can get a close approximation for the blur. This can be written as,

$$LR[H_p] = LR[I_p + B_p] \simeq B_p \quad (5)$$

where LR represents low rank approximation using SVD. It should be emphasized that only the first few singular vectors will characterize blur functions whereas remaining will represent image. For estimation of any blur, only a few singular vectors are sufficient. Blur parameters can then be estimated from exponent of (5) by finding location of zeros (for uniform motion or out of focus blur) or by fitting appropriate Gaussian (for Gaussian blur). In order to have a more reliable estimate of the blur parameters, we use several (candidates of) blur estimates obtained from single image. It should be noted that when the degraded image has additive noise, the low rank approximation will characterize blur as well as added noise.

### 3.1 Gaussian Blur

When an image is blurred using Gaussian blur, the parameters of Gaussian can be estimated from low rank approximation as described above. These parameters can be derived from a scaled 2D version of Gaussian blur ( $B_{m,n}$ ) in Table 1 as below:

$$B_{m,n} = \frac{\alpha}{2\pi\sigma^2} e^{-\frac{(m^2+n^2)}{2\sigma^2}} \quad (6)$$

Where  $m$  and  $n$  are Fourier counter-parts of  $x$  and  $y$ .

If  $\sigma$  is defined as blur parameter and  $\alpha$  as the scale factor, Blur parameters ( $\sigma$ ) can be calculated by

$$\frac{\frac{m^2+n^2}{2}}{-1 \pm \sqrt{1 - (\frac{m^2+n^2}{2})(-\log(B_{m,n}) - \log(\alpha) - \log(2\pi))}} \quad (7)$$

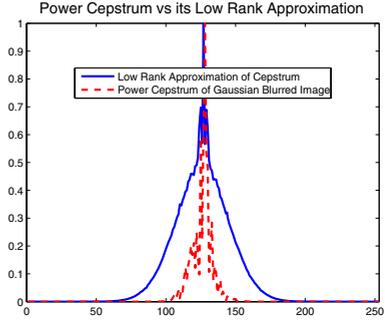
A complete proof of (7) is provided in Appendix. Once blur estimates are obtained from (5), we can directly find the parameters using (7) for noiseless image degradation. The mode of the distribution of candidate parameters is used to calculate the final blur parameters.

When the blurred image has additive noise, it can be suppressed by integrating the noisy blurred image in one direction. The parameters for blur ( $\sigma$ ) in this case can be estimated using (8). A complete proof of (8) is also provided in Appendix. The parameters for blur in noisy case, are given by,

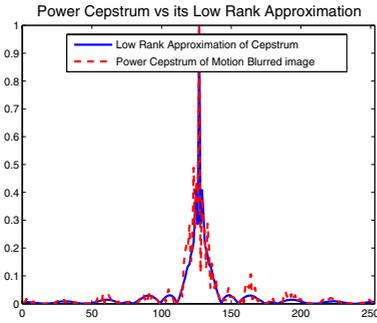
$$\frac{n^2}{1 \pm \sqrt{1 + (2n^2)(-\log(B_m) - \log(\alpha) - 1 - \frac{\log(2\pi)}{2})}} \quad (8)$$



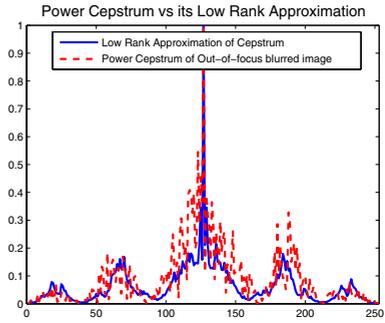
(a)



(b)



(c)



(d)

**Fig. 1.** Comparison of Blur Estimation using Cepstrum vs its Low Rank approximation when input image (a) is degraded by blur type: (b) Gaussian, (c) Linear Motion, and (d) Out of focus

For noisy Gaussian blur, low rank approximation characterizes both blur as well as noise. Since the blurred image had additive noise, the parameters estimated will be noisy. Therefore, it is reasonable to assume that the correct blur parameters can be estimated by fitting Gaussian mixture model [10].

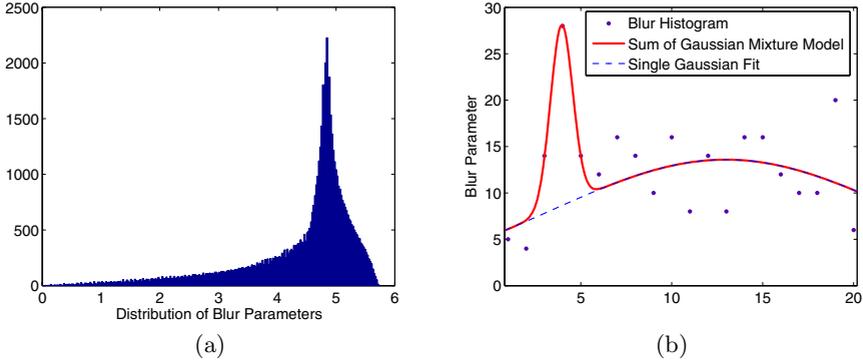
### 3.2 Motion Blur

When an image is blurred with motion blur, blur parameters for motion blur are calculated using (9). These parameters are derived by finding the location of zeros of the blur estimates as below:

$$\frac{\sin(\pi a f_x)}{\pi a f_x} = \sin(\pi k) = 0 = \pi a f_x = \pi k$$

$$\implies f_x a = k \tag{9}$$

where  $\mathbf{a}$  is the parameter to be estimated,  $\mathbf{k}$  is the number of zero crossings whereas  $\mathbf{f}_x$  relates to the location of zero crossing. A reliable estimate of blur parameter can be found using a system as given below:



**Fig. 2.** Example of Gaussian Blur Estimation: (a) Distribution of all blur parameters, and (b) Maximum-likelihood estimate of parameters by fitting Gaussian mixture model

$$\mathbf{F}_x \cdot \mathbf{a} = \mathbf{K} \quad (10)$$

where  $\mathbf{F}_x$  and  $\mathbf{K}$  are matrices with several values of  $\mathbf{f}_x$  and  $\mathbf{k}$  from (9) stacked respectively together.

### 3.3 Out of Focus Blur

When an image is blurred with out of focus blur, blur parameters are calculated in a manner similar to (9) where a least square system for out of focus blur can be created as in (10). These parameters are derived by finding the location of zeros of the blur estimates as below:

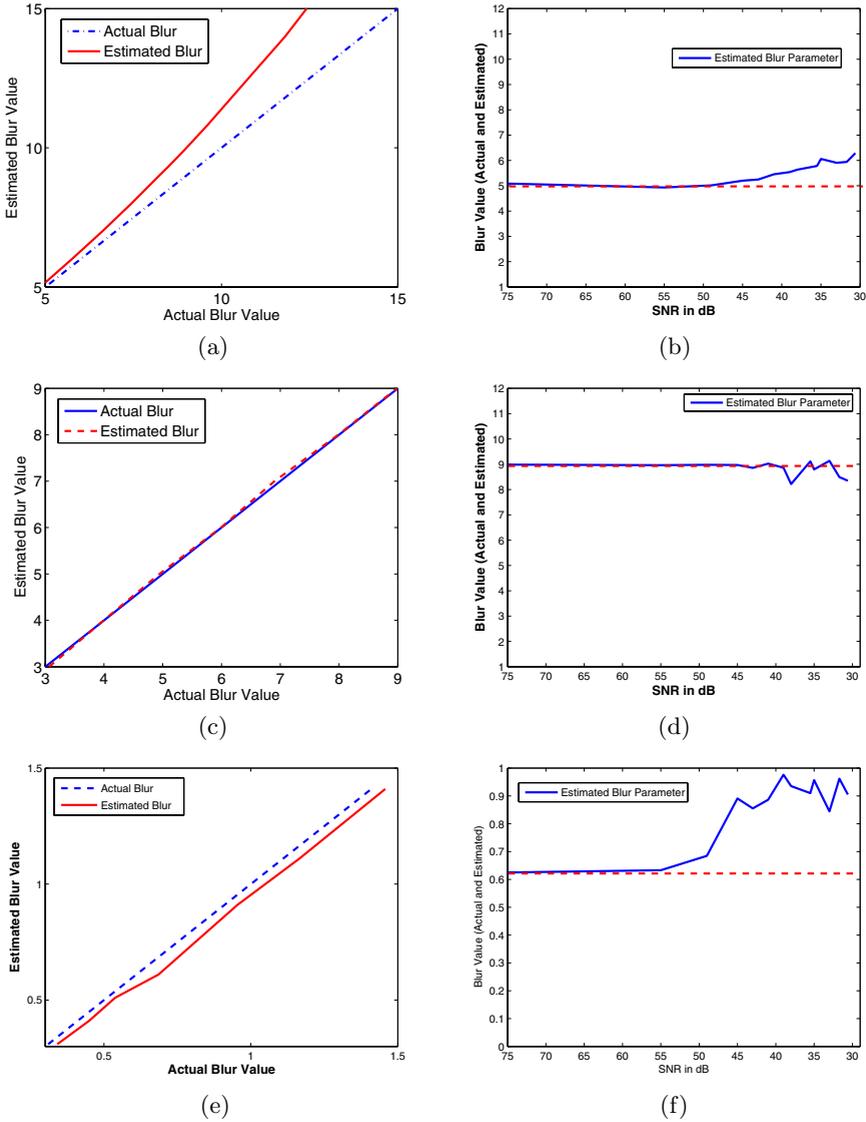
$$\frac{J_1(2\pi a f)}{\pi a f} = 0 \implies r f = k$$

where  $\mathbf{r}$  is the parameter to be estimated,  $\mathbf{k}$  is the number of zero crossings, and  $\mathbf{f}$  is related to the location of zero crossings. It should be mentioned that in our results, we have used the value of  $\mathbf{f}$  as provided in [3].

## 4 Results

The proposed method can be used to estimate any type of blur function but results for Gaussian, uniform motion, and out of focus blurs have been presented here. Once low rank approximation of cepstrum is obtained, initial blur parameters are estimated followed by their robust estimates (from several candidates). We have presented the results both in the noiseless and noisy cases (with different signal-to-noise ratios).

Figure 1(a) shows the original image before any degradation. Figure 1(b) shows the comparison of Gaussian blur parameter estimation for cepstrum verses the low rank approximation of cepstrum. It is clearly obvious that the low rank



**Fig. 3.** Comparison of actual vs estimated blur parameters when blur type is: (a) Noiseless Gaussian Blur, (b) Noisy Gaussian Blur, (c) Noiseless Motion Blur, (d) Noisy Motion Blur, (e) Noiseless Out of Focus Blur, (f) Noisy Out of Focus Blur

(approximation) version is less erratic and easy to use. The observation holds true for Linear Motion Figure 1(c) and Out of focus Figure 1(d) blurs. Using several slices of our blur estimates, we can find the parameters using (7). The mode of the distribution of these candidate parameters, shown in Figure 2(a), is used to calculate the final blur parameter. The comparison of actual and estimated

blur values has been shown in Figure 3(a). For noisy Gaussian blur, the low rank approximations have both noise as well as blur parameters. In order to estimate and subsequently separate both distributions, Gaussian curve fitting is used. Figure 2(b) shows the fitting of Gaussian mixture model on the distribution of noisy parameters. Since the noise usually has small mean value, the mode of larger distribution gives us the correct estimate of blur parameters. The comparison of actual and estimated blur values has been shown in Figure 3(b). It should be noted that, for the noisy case, results for average values of Gaussian blur estimates over 100 independent trials have been plotted against the actual values under different noise levels.

Next, an image was blurred using uniform motion blur in one direction, as proposed in [3], and blur parameters were estimated using low rank approximation of cepstrum. These motion blur parameters are calculated using (9) after ‘properly’ thresholding the low rank approximations. It should be emphasized that motion blur is only in one direction therefore, several estimates of blur parameters are available along the direction orthogonal to the motion. To estimate robust parameters having consensus over all available estimates, we build a over-determined system of equations and find the least square solution as in (10). The comparison of actual and estimated values of motion blur using our method is shown in Figure 3(c) and (d) for noiseless and noisy cases respectively. It should again be noted that in the noisy case, results for average values of motion blur estimates over 100 independent trials have been plotted against actual values under different noise levels. Similarly, blur parameters for images blurred by out of focus blur are estimated and are shown in Figure 3(e) and (f) both for noiseless and noisy cases respectively.

#### 4.1 Discussion

Most Fourier transform based techniques require high resolution of input images [8]. Their performance also reduces when the blur amount is too high or signal-to-noise ratio is too low [1]. We have presented a technique which does not require high resolution images. The images used in this paper were of smaller size 253x253 as compared to the resolution required by [8] i.e., 512x512. Moreover, we are able to estimate the blur when the blur value is high and signal-to-noise ratio is low. Figure 3 shows the results for a wide range of SNR ratios. It should also be emphasized that our technique uses low rank approximation of cepstrum and therefore requires only fewer singular vectors for blur estimation as compared with any other known technique.

## 5 Conclusion

We have presented a novel technique for blind blur estimation using low rank constraints on blur functions. The main idea this paper presents is that the blur functions usually have low ranks when compared with images, allowing us to estimate them robustly from a single image. We have also quantified the exact rank that should be used for different blur types. One major strength of our

approach is that it presents a general framework for low rank approximation since rank constraints apply to all blur functions in practice. Results for Gaussian, motion, and out of focus blurs are presented to show that the technique works for most common blur types and at various noise levels. It has also been shown that the technique does not require any assumptions on imaging system and works well for low resolution images.

## References

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## Appendix

### 1: Proof of Equation (7)

Taking logarithm of (6) we get,

$$\log(B_{m,n}) = \log(\alpha) - \log(2\pi\sigma^2) + \frac{-m^2}{2\sigma^2} + \frac{-n^2}{2\sigma^2} \quad (11)$$

which can be written as,

$$\log(B_{m,n}) = \frac{1}{\sigma^2} \left( \frac{-m^2 - n^2}{2} \right) + \log(2) + 2\log\left(\frac{1}{\sigma}\right) - \log(\alpha) - \log(2\pi)$$

or

$$\log(B_{m,n}) = \frac{1}{\sigma^2} \left( \frac{-m^2 - n^2}{2} \right) + 2\log\left(1 + \frac{1}{\sigma}\right) - \log(\alpha) - \log(2\pi)$$

If  $\sigma$  and  $\alpha$  are constants, the 1<sup>st</sup> order Taylor series approximation of above equation gives,

$$\frac{1}{\sigma^2} \left( \frac{-m^2 - n^2}{2} \right) + 2 \left( \frac{1}{\sigma} \right) - \log(B_{m,n}) - \log(\alpha) - \log(2\pi) = 0$$

hence,

$$\frac{1}{\sigma} = \frac{-1 \pm \sqrt{1 - \left( \frac{m^2+n^2}{2} \right) (-\log(B) - \log(\alpha) - \log(2\pi))}}{\frac{m^2+n^2}{2}} \quad (12)$$

Therefore,  $(\sigma)$  for noiseless case is given by,

$$\frac{\frac{m^2+n^2}{2}}{-1 \pm \sqrt{1 - \left( \frac{m^2+n^2}{2} \right) (-\log(B_{m,n}) - \log(\alpha) - \log(2\pi))}} \quad (13)$$

## 2: Proof of Equation (8)

Integrating (6) along the direction orthogonal to motion and taking logarithm gives,

$$\log(B_x) = \log(\alpha) - \frac{1}{2} \log(2\pi) + \log\left(\frac{1}{\sigma}\right) - \frac{y^2}{2\sigma^2} \quad (14)$$

If  $\sigma$  and  $\alpha$  are constants, the 1<sup>st</sup> order Taylor series approximation of above equation gives,

$$\frac{1}{\sigma^2} \left( \frac{-y^2}{2} \right) - 1 + \frac{1}{\sigma} - \log(B_{m,n}) + \log(\alpha) - \frac{1}{2} \log(2\pi) = 0$$

hence,

$$\frac{1}{\sigma} = \frac{-1 \pm \sqrt{1 - 2y^2(-\log(B_x) + \log(\alpha) + (\frac{1}{2})\log(2\pi) - 1)}}{-y^2} \quad (15)$$

Therefore,  $(\sigma)$  for noisy case is given by,

$$\frac{-y^2}{-1 \pm \sqrt{1 - 2y^2(-\log(B_x) + \log(\alpha) + (\frac{1}{2})\log(2\pi) - 1)}} \quad (16)$$