$\lambda_1 = 0.9$, and $\lambda_2 = \lambda_3 = 0.05$. The covariance matrices $\mathbf{R_1}$ and $\mathbf{R_2}$ are the same as the previous example, and

$$\mathbf{R_3} = \begin{bmatrix} 200 & 60+60j \\ 60-60j & 150 \end{bmatrix}.$$

The robustness of our adaptive receiver is further tested in spherically invariant random process (SIRP) noise and sub-Gaussian α-stable noise. For SIRP noise, we chose the Laplace $(b = 1)$, K-distribution ($b = 1$, $\nu = 0.5$), and Student-t ($b = 1$, $\nu = 1$) distribution from Table I of [4]. Both the SIRP and sub-Gaussian noise use $\mathbf{R_1}$ as the underlying covariance matrix. Other parameters are $N = 2$, $m = 1$, and $\mathbf{S_1} = -\mathbf{S_2} = [1]$. Performance is given in Tables IV and V, respectively. All the simulation results are based on 500 runs with each run involving 1000 bits. In all the examples, performance of the optimum receiver using the true parameters is also given for comparison. For sub-Gaussian noise, the receiver suggested in [5] is used as a comparison. It can be seen that in all cases, performance improves as T increases. Near-optimum performance is achieved when enough training data is available ($T = 1000$). The robustness of our adaptive receiver is easily explained by the fact that Gaussian mixture densities with enough mixing terms can approximate many non-Gaussian distributions with high accuracy.

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Noise Suppression by Removing Singularities

H. Shekarforoush

***Abstract*—A method is proposed for suppressing Gaussian noise by extracting local singularities. The method is based on truncating Riemann series decomposition, whose components naturally characterize different orders of Hölder regularity. The approach yields a single-step filtering technique whose performance is comparable to three-step wavelet decomposition and thresholding techniques.**

***Index Terms*—Fractional singularities, Hölder exponents, noise suppression.**

I. INTRODUCTION

The literature on noise estimation and suppression is remarkably rich. This makes the classification and comparison of the existing methods somewhat difficult. Classical linear and nonlinear methods can be broadly divided into two main categories of statistical filtering techniques and Fourier-domain methods. Statistical filtering techniques include the L-estimators [16], [19], [26], the M-estimators [10], [9], and the more recent techniques based on higher order statistics [20], [22], [25]. The Fourier domain methods are based on the extraction of regularity by imposing some growth conditions on the signal, which reduces the space of admissible solutions to functions of given type and order, e.g., entire functions of exponential type [8], [21], [28], functions of polynomial growth [2], [11], [15], etc. In these methods, regularity is defined in a global sense based on Riemann's lemma (see [13, pp. 95–96]). Noise removal is then achieved by a filtering technique assuming a different rate of convergence for the Fourier coefficients of the regular and the irregular portions of the noisy signal.

Recent efforts on the subject have been mostly divided into two lines of research. One is based on modeling signal/image denoising as a diffusion process [4], [5], [14], [17], [18], [27], and the other is based on the concept of removing local singularities using wavelet-based multiscale techniques [1], [6], [7], [12], [23], assuming that the singularities are relatively isolated.

In wavelet-based multiscale techniques regularity/singularity is described locally using Hölder exponents (to be defined in the next section). Wavelet-based multiscale methods propose a three step approach:

2) decomposition;
3) characterization followed by some sort of thresholding;
4) reconstruction.

Our motivation in this work is to use the notion of local Hölder regularity but combine these three steps into a single step in order to reduce the problem to that of filtering the noisy signal, as with most other existing methods. An obvious approach would be to find a method that decomposes signals directly in terms of components that automatically separate different orders of singularity. In the following section, we set the theoretical background and derive some analytic expressions for this purpose. This will be followed by the implementation and the results, with some concluding remarks at the end.

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II. Removing Isolated Singularities

Our aim in this section is to show that under some signal regularity conditions, isolated singularities (due to for instance noise) can be removed by filtering. We first consider integer order singularities and then generalize the results to noninteger orders. The order of a singularity is given by its Hölder exponent α defined as follows.

Definition II.1: A function $f(t)$ is said to be Lipschitz or Hölderian of order α at t_0 $(\alpha \in]0,1])$ if for an arbitrarily small ϵ there exists a positive finite constant c such that

$$|f(t_0) - f(t_0+\epsilon)| \le c|\epsilon|^{\alpha}. \tag{1}$$

It follows that for $\alpha \in]0,1[$, the function is singular at t_0, and for $\alpha = 1$, it is differentiable at t_0. Higher order regularities (and lower order singularities) can be defined by $k < \alpha \le k+1$, where $k \ne 0$ is an integer.

A singularity is called of integer order if its Hölder exponent is an integer and is said to be isolated if the following definition applies.

Definition II.2: If f on $\mathbb{C}$ is analytic everywhere on a disc $D = \{z : 0 \le |z - z_0| \le A_1\}$, except at $z = z_0$, then z_0 is said to be an isolated singularity of f, and $D - \{z_0\}$ is said to be a deleted region of z_0.

Here, analyticity is to be understood in terms of Cauchy's integral theorem [24, p. 75]. We will show below that isolated integer-order singularities can be removed by extracting the principal part (i.e., the singular part) of the Laurent series, provided that f satisfies some regularity conditions. The principal part of the Laurent series of f is given by

$$f_p(z) = \sum_{r=1}^{\infty} a_{-r}(z-z_0)^{-r}. \tag{2}$$

The following theorem gives our first result.

Theorem II.1: Let $f(z)$ be a function defined on $\mathbb{C}$ with a set of (possibly denumerable) isolated integer order singularities. In addition, let the nonsingular portion of f be bandlimited and absolutely Lebesgue integrable. The Fourier transform of the principal part of Laurent series expansion of $f(z)$ at any singular point satisfies

$$\hat{f}_p(u) = \hat{f}(u)\left(\frac{1}{iu-1}\right) \tag{3}$$

where the hat sign stands for the Fourier operator.

See the Appendix for proof.

As discussed later, in general, singularities of signals and noise are not limited to integer orders. Therefore, since the filter $1/(iu-1)$ cannot extract noninteger order singularities, the above result is of no practical interest in signal processing. However, a natural extension can be found by using the Riemann series expansion as follows:

$$f(z) = \sum_{r=-\infty}^{\infty} a_r(z-z_0)^{r+\alpha} \tag{4}$$

where

$$a_r = \frac{\mathcal{D}_{z-b}^{r+\alpha} f(z)}{\Gamma(1+r+\alpha)} \tag{5}$$

where

- $\mathcal{D}_{z-b}^{r+\alpha}$ fractional derivative operator of order $r+\alpha$ with respect to $z-b$;
- α arbitrary complex number;
- Γ Euler's gamma function.

The fractional derivative of f is given by Cauchy's integral formula taken around any closed rectifiable Jordan curve

$$\mathcal{D}_{z-b}^{r+\alpha} f(z) = \frac{\Gamma(1+r+\alpha)}{2\pi i} \oint_{\Upsilon} \frac{f(z)\,dz}{(z-z_0)^{1+r+\alpha}}. \tag{6}$$

We will assume, hereafter, that $\Im m\{\alpha\} = 0$. The next theorem generalizes our result to noninteger order singularities.

Theorem II.2: Let $f(z)$ be a function defined on $\mathbb{C}$ with a set of (possibly denumerable) isolated noninteger or integer order singularities. In addition, let the nonsingular portion of f be bandlimited and absolutely Lebesgue integrable. Then, the Fourier transform of the principal part of its Riemann series expansion satisfies

$$\hat{f}_p(u) = \lim_{\alpha \downarrow -1} \hat{f}(u)\hat{\mathcal{K}}(u) \tag{7}$$

$$\hat{\mathcal{K}}(u) = \left(\frac{u^{-\alpha}}{1+u^2}\left(u\exp(-i\theta) - \exp\left(i\frac{\pi}{2} - i\theta\right)\right)\right) \tag{8}$$

where $\theta = (\pi/2)(1+\alpha)\,\mathrm{sgn}(u)$, and sgn is the signum function.

See the Appendix for proof.

This result shows that the spectrum of the singular portion of f can be factorized as the product of the spectrum of f and that of a filter given by (8). This is equivalent to first expanding a signal in a series whose terms are in an ascending order of Hölder exponents and then removing the unwanted singular terms by truncation.

III. Implementation and Results

To apply the results for denoising, we need to consider two issues. First recall that the Gaussian noise is a nowhere differentiable continuous function whose Hölder exponents are uniformly negative [12], whereas regular functions are uniformly positive Hölderian. Secondly, we have

$$\lim_{\alpha \downarrow -1} \hat{\mathcal{K}}(u) = \frac{u}{1+u^2}(u-i) \tag{9}$$

Therefore, for $\alpha \downarrow -1$ the following filter will extract the regular portion of f:

$$\lim_{\alpha \downarrow -1} 1 - \hat{\mathcal{K}}(u) = \frac{1}{1+u^2} - iu\frac{1}{1+u^2} \tag{10}$$

Here, $1/(1+u^2)$ is the spectrum of $(1/2)\exp(-|z|)$, which is not differentiable at the origin. However, the second term in (10) can exist in Cauchy principal value sense, in which case, the impulse response $h(z)$ of the filter in (10) will converge to a right-sided exponential, which is separable and is real when z is restricted to the real line. Therefore, for real signals, if we for instance require a zero-phase response, we can apply the following separable filter:

$$\hat{h}(u) = \left|\frac{1-iu}{1+u^2}\right| = \frac{1}{\sqrt{1+u^2}} \tag{11}$$

which can also be frequency scaled in order to take into account bandwidth variations

$$\hat{h}(u) = \frac{1}{\sqrt{1+(su)^2}} \quad \text{where } s \in \mathbb{R}^+. \tag{12}$$

Many simulations for one– and two–dimensional (1-D and 2-D) signals have been performed, some of which are given in Figs. 2–6. Results have been compared to denoising by soft thresholding using Daubechies' symlet wavelet bases (see [3, pp. 251–287]). Simulations show that our filter has an excellent performance comparable to wavelet-based techniques. Extensive experiments have also been performed with classical filters, e.g., moving average and Wiener filtering, which due to lack of space have not been presented here. Results show the superiority of our filter to classical filters, particularly in low SNR.

IV. Conclusion

In this correspondence, we have shown that removing Gaussian noise singularities can be achieved by simple filtering. The filter is

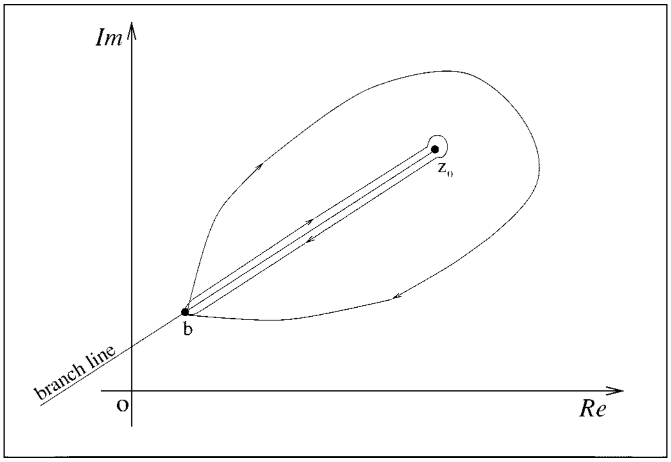


Fig. 1. Deformation of the contour of integration in Cauchy's formula.

obtained by expanding the noisy signal in Riemann series and then truncating it to the regular portion. The spectrum of the truncated version will then factorize into the product of the spectrum of the original signal and that of a denoising filter.

The advantage of this approach to other decompositions, such as wavelet transforms, is that in Riemann series, each term naturally characterizes a different order of Hölder regularity. On the other hand, since these terms are organized in ascending order of regularity, a simple truncation can separate the singularities.

However, in order to obtain a closed-form solution, we made the analyticity assumption, which may affect the performance of the filter. Nevertheless, the results demonstrate an excellent performance, particularly when the SNR is extremely low. At low/medium SNR, the proposed filter outperforms classical filters and has a performance comparable with wavelet-based techniques.

APPENDIX

Proof of Theorem II.1: We first deform the contour of integration into a line joining an isolated singularity z_0 and an arbitrary branch point b in the analytic region around z_0 (see Fig. 1).

$$a_{-r} = \int_b^{z_0} \frac{f(z)\,dz}{(z-z_0)^{1-r}} \quad \text{where } r \geq 1 \tag{13}$$

We will evaluate (13) in the Fourier domain. For this purpose, note that if $\Re e(b) > 0$ and $r \geq 1$, then

$$(iu)^{-r} = \lim_{b\to 0} \frac{1}{(r-1)!} \int_{-\infty}^{\infty} \frac{\hbar(z)\exp(-bz)}{z^{1-r}} \times \exp(-izu)\,dz \tag{14}$$

where $\hbar(\cdot)$ is the Heaviside function applied to the magnitude of its argument.

Therefore

$$\hat{a}_{-r} = (r-1)!(iu)^{-r}\hat{f}(u). \tag{15}$$

Let $b_{-r} = (z-z_0)^{-r}$. By Fourier transforming both sides, we get for all $r \geq 1$

$$\hat{b}_{-r} = \exp(-iuz_0)(2\hbar(u)-1)\frac{(-1)^r(iu)^{r-1}}{(r-1)!} \tag{16}$$

and hence

$$\hat{f}_p(u) = \sum_{r=1}^{\infty} \hat{a}_{-r} * \hat{b}_{-r} = \sum_{r=1}^{\infty} \left(\frac{\hat{a}_{-r}}{(r-1)!}\right) * (\hat{b}_{-r}(r-1)!) \tag{17}$$

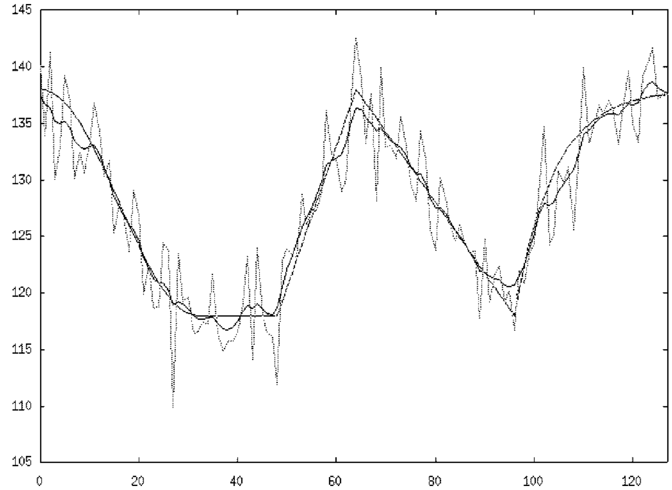


Fig. 2. Test signal, noisy signal (SNR $\simeq$ 7 dB) and the restored signal (SNR $\simeq$ 17 dB).

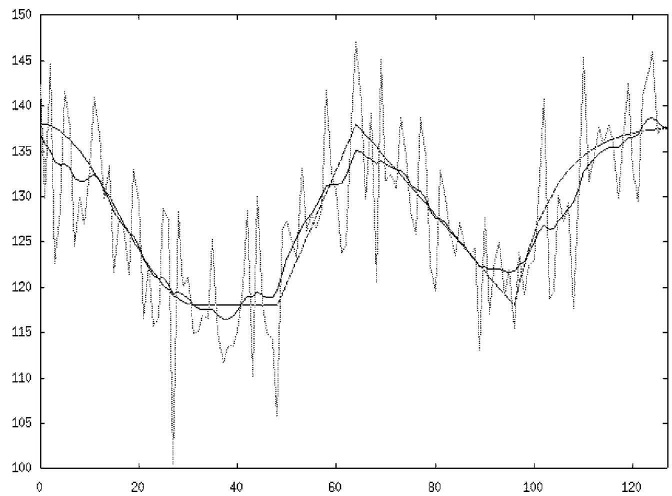


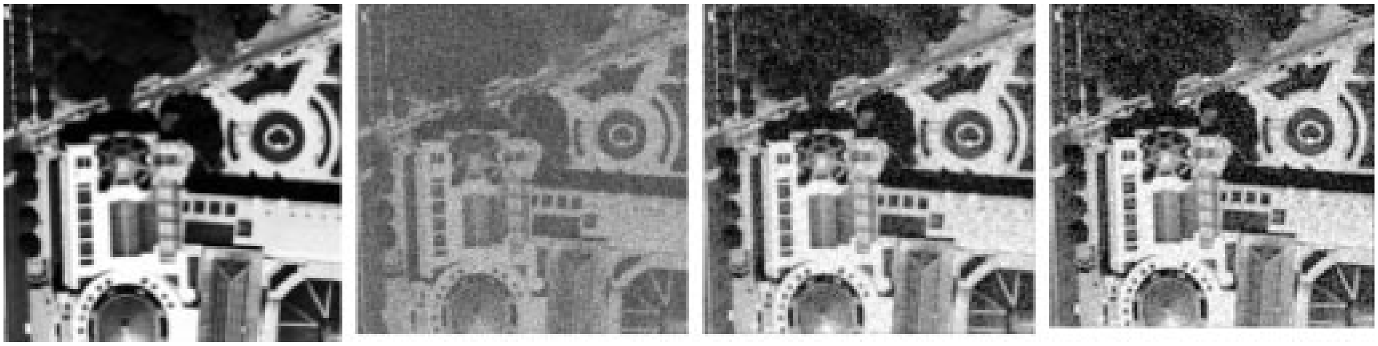
Fig. 3. Test signal, noisy signal (SNR $\simeq$ 1 dB) and the restored signal (SNR $\simeq$ 13 dB).

or equivalently

$$\hat{f}_p(u) = \left(\sum_{r=1}^{\infty} \frac{\hat{a}_{-r}}{(r-1)!}\right) * \left(\sum_{r'=1}^{\infty} \hat{b}_{-r'}(r'-1)!\right) - \sum_{r'\neq r} \hat{a}_{-r} * \hat{b}_{-r'}. \tag{18}$$

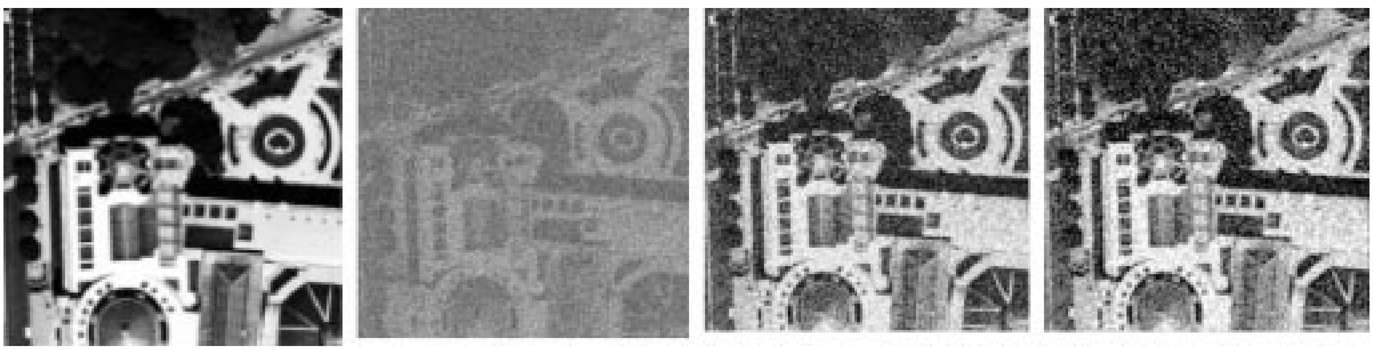
The last term can be simplified as follows:

$$\begin{aligned}
\sum_{r'\neq r} \hat{a}_{-r} * \hat{b}_{-r'} &= \sum_{r'\neq r} ((iu)^{-r}\hat{f}(u)) * (\exp(-iuz_0) \\
&\quad \times (2\hbar(u)-1)(-1)^{r'}(iu)^{r'-1}) \\
&= \sum_{r'\neq r} \int_{-\sigma}^{\sigma} ((iu)^{-r}\hat{f}(u))(\exp(-i(u-u_0)z_0) \\
&\quad \times (2\hbar(u-u_0)-1)(-1)^{r'}(iu-iu_0)^{r'-1})\,du \\
&= \sum_{r'\neq r} \int_{-\sigma}^{\sigma} ((iu)^{-r}\hat{f}(u)) \Bigg(\exp(-i(u-u_0)z_0)
\end{aligned}$$



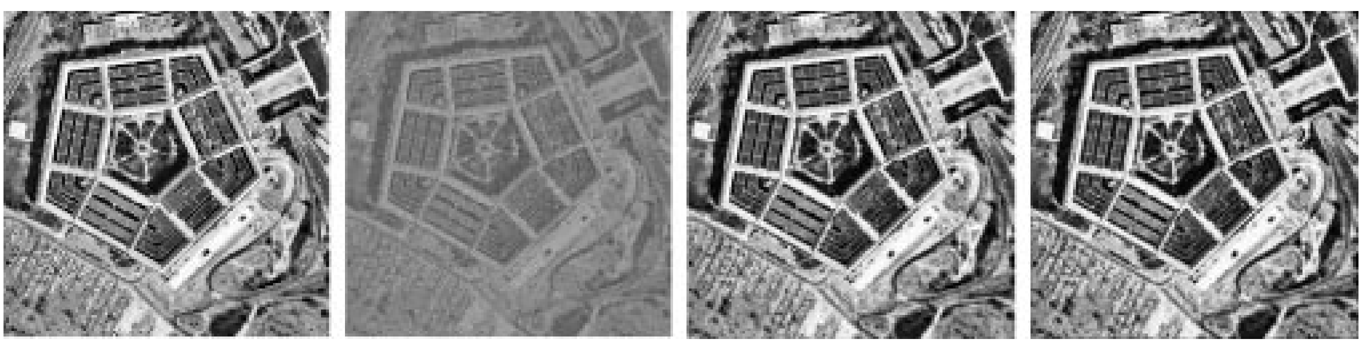
(a) (b) (c) (d)

Fig. 4. (a) Test image. (b) Noisy observation (SNR $\simeq 2.6$ dB). (c) Our filter (SNR $\simeq 11.5$ dB). (d) Soft thresholding using a Daubechies' symlet wavelet (SNR $\simeq 11.2$ dB).



(a) (b) (c) (d)

Fig. 5. (a) Test image. (b) Noisy observation (SNR $\simeq -2$ dB). (c) Our method (SNR $\simeq 8.5$ dB). (d) Soft thresholding using a Daubechies' symlet wavelet (SNR $\simeq 9$ dB).



(a) (b) (c) (d)

Fig. 6. (a) Test image. (b) Noisy observation (SNR $\simeq 6$ dB). (c) Our filter (SNR $\simeq 9$ dB). (d) Soft thresholding using a Daubechies' symlet wavelet (SNR $\simeq 8.6$ dB).

$$
\begin{aligned}
&\times (2\hbar(u-u_0)-1)(-1)^{r'}\sum_{k=0}^{r'-1} {}^{r}c_k (iu)^k \\
&\times \left.(iu_0)^{r'-k-1}\right) du \\
&= \sum_{r'\neq r}\int_{-\sigma}^{\sigma}\hat{f}(u)\Biggl(\exp(-i(u-u_0)z_0) \\
&\times (2\hbar(u-u_0)-1)(-1)^{r'}\sum_{k=0}^{r'-1} {}^{r}c_k (iu)^{-r+k} \\
&\times \left.(iu_0)^{r'-k-1}\right) du \\
&= \sum_{r'\neq r}\hat{f}(u) * (\exp(-iuz_0) \\
&\times (2\hbar(u)-1)(-1)^{r'}(iu)^{r'-r-1}) \\
&= \hat{f}(u) * \sum_{r'>r}(\exp(-iuz_0) \\
&\times (2\hbar(u)-1)(-1)^{r'}(iu)^{r'-r-1}) \\
&+ \hat{f}(u) * \sum_{r'<r}(\exp(-iuz_0) \\
&\times (2\hbar(u)-1)(-1)^{r'}(iu)^{r'-r-1})
\end{aligned}
$$

where ${}^{r}c_k$ are the binomial coefficients, and u_0 is an arbitrary shift.

Since both terms converge to zero, we get

$$\sum_{r'\neq r}\hat{a}_{-r} * \hat{b}_{-r'} = 0. \tag{19}$$

Now, let $\hat{d}_{-r'} = \sum_{r'=1}^{\infty} \hat{b}_{-r'}(r'-1)$. Hence

$$d_{-r'} = \hbar(z-z_0)\exp(z-z_0) \tag{20}$$

which tends to one as $|z-z_0| \downarrow 0$.

On substituting (15), (19), and (20) into (18) and simplifying, we get

$$\hat{f}_P(u)|_{z=z_0} = \hat{f}(u)\sum_{r=1}^{\infty}(iu)^{-r} \tag{21}$$

$$= \hat{f}(u)\left(\frac{1}{iu-1}\right). \tag{22}$$

Proof of Theorem II.2: To prove this theorem, simply replace every factorial by the gamma function and r by $r+\alpha$; then, the above analysis will still strictly hold if we assume that $\Re e\{r+\alpha\} > 0$. ■

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